

Frobenius Endomorphisms for Koszul Complexes

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AMS Spring Eastern Sectional Meeting

- 1 Rings of Prime Characteristic
- 2 Homological Tools
- 3 Upgrading to Koszul Complexes

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Such rings are equipped with a Frobenius endomorphism

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- $\sqrt[p]{x} + \sqrt[p]{y} = \sqrt[p]{x+y}$ and $\sqrt[p]{x} \cdot \sqrt[p]{y} = \sqrt[p]{xy}$, and
- $r \cdot \sqrt[p]{x} = \sqrt[p]{r^p x}$ for all $r \in R$.

Example

Let $k = \mathbb{Z}/p\mathbb{Z}$ and let $R = k[x]$. Then F_*R is generated by $a = \sqrt[p]{x}$ satisfying $a^p = x$ showing

$$F_*R \cong \frac{R[a]}{(a^p - x)} = \bigoplus_{i=0}^{p-1} Ra^i$$

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For $R = k[x]/(x^2)$ $(\sqrt[p]{x})^2 = \sqrt[p]{x^2} = 0$ showing $a = \sqrt[p]{x}$ satisfies $a^2 = 0$ and $a^p = x$. Hence

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We say R is *F-finite* if F_*R is f.g. as an R -module.

For F-finite local rings, the homological data of F_*R classifies the regularity of the ring.

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Question

How does F_*R encode the homological data for every module?

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Definition

Let M and N be two R -modules. The level of M against N , denoted $\text{level}_R^N(M)$, is the smallest integer d such that there is a resolution $X \xrightarrow{\sim} M$ of the form

$$0 \rightarrow X_{d-1} \rightarrow X_{d-2} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

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We write $N \Rightarrow M$ if $\text{level}_R^N(M) < \infty$.

We say N is a generator if $N \Rightarrow M$ for all f.g. R -modules M .

Example

Let R be a Noetherian ring and M a f.g. R -module.

$$R \Rightarrow M \Leftrightarrow \begin{array}{l} M \text{ has a finite length} \\ \text{projective resolution} \end{array} \Leftrightarrow \text{pd}_R(M) < \infty$$

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Heuristic

Let Hdim be a homological dimension and let M, N be two complexes with $N \Rightarrow M$.

If $\text{Hdim}(N) < \infty$ then $\text{Hdim}(M) < \infty$.

Theorem (Ballard-Iyengar-Lank-Mukhopadhyay-Pollitz '25)

Let R be an F -finite ring, then $F_^e R$ is a generator for $e \gg 0$. Moreover if R is a locally complete intersection then the first pushforward $F_* R$ is a generator.*

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1. A local-to-global principal from Benson-Iyengar-Krause '15:

$$F_*^e R \text{ is a generator} \iff \begin{array}{l} F_*^e R_{\mathfrak{p}} \Rightarrow \kappa(\mathfrak{p}) \\ \text{for all } \mathfrak{p} \in \operatorname{Spec}(R) \end{array}$$

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- 2a. In the general case the level is bounded above by $\operatorname{edim}(R_{\mathfrak{p}}) + 1$ using simplicial algebras.
- 2b. In the locally complete intersection case cohomological supports and a theorem of Stevenson '14 show the level is finite without constructing an upper bound.

Question

Is $\text{level}_R^{F_* R}(k) \leq \text{edim}(R) + 1$ for an F-finite local complete intersection?

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Definition

Let $f \in R$, then the Koszul complex on f over R is the complex

$$K^R(f): Re \xrightarrow{f} R1$$

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When $f \in R$ is a non-zerodivisor, then $K^R(f) \xrightarrow[\cong]{R} R/f = S$.

Quasi-isomorphisms preserve homological data, so we can work with (DG) modules over $K^R(f)$ instead of over S .

Definition (- '26)

The Frobenius endomorphism for $E = K^R(f)$ is the chain map $F^E: E \rightarrow E$ given by

$$\begin{array}{ccc} Re & \xrightarrow{f} & R1 \\ \downarrow f^{p-1} \cdot F^R & & \downarrow F^R \\ Re & \xrightarrow{f} & R1 \end{array}$$

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Letting $A = F_*R$, the Frobenius pushforward of E is the complex $F_*E = K^A(\sqrt[p]{f})$

$$F_*E: A \sqrt[p]{e} \xrightarrow{\sqrt[p]{f}} A1$$

The E -algebra structure is given by $e \cdot (-) = \sqrt[p]{f^{p-1}} \cdot \sqrt[p]{e} \cdot (-)$.

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Letting $E = K^R(f)$ and $A = F_*R$ we see

$$F_*E: A\sqrt[3]{e} \xrightarrow{a^2} A1$$

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Representing the differential and multiplication as a matrix

$$\partial = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 1 & 0 & 0 \end{bmatrix} \quad e \cdot (-) = \begin{bmatrix} 0 & 0 & x^2 \\ x & 0 & 0 \\ 0 & x & 0 \end{bmatrix}$$

Theorem (- '26)

Let $(R, \mathfrak{m}, k)$ be an F -finite regular local ring, $f \in R$, and $E = K^R(f)$. Then $\text{level}_E^{F_*E}(k) \leq \text{edim}(R) + 1$ and F_*E is a generator.

Corollary (-, '26)

Let R be an F -finite ring which is locally hypersurface, then

$$\text{level}_{R_{\mathfrak{p}}}^{F_*R_{\mathfrak{p}}}(\kappa(\mathfrak{p})) \leq \text{edim}(R) + 1 < \infty \text{ for all } \mathfrak{p} \in \text{Spec}(R)$$

Thank you!

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